3-D ARCHERY
A GUIDE TO COURSE DESIGN

ANNEX B - FORMULAE AND TABLES

One premise on which this Guide has been based is that the danger area for a bow can be determined by a combination of calculation, intuition and experience. I promised early in the Guide that I would relegate the math to the back of the book and that is where we are now. For those who are ready to accept my calculations at face value, I thank you for your trust and you can skip this section. For the doubtful, the curious and those with an urge to check my math, please, read on. If you can refine these figures, I look forward to your comments.

In the development of the course safety standards I generated figures for the following;

- Distances to ground impact for the standard shot for arrow velocities of 180, 240, and 300 feet per second (ft/s);
- Distances to ground impact for the standard shot velocities when the ground behind the target falls away; and,
- Ranges for the standard shot velocities for selected upward angles of fire.

For each of these I will present the equations I began with, those I derived and tables of calculations.

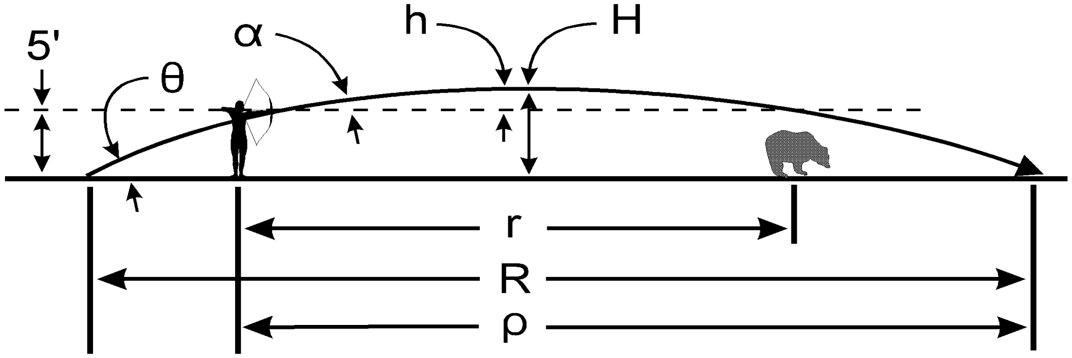
<u>The Standard Shot.</u> Prior to working through the calculations for range data, it will be useful if we restate the conditions for the standard shot. All data except that for ranges based on upward angles of departure were developed with these conditions as a starting point. Figure 20 illustrates the variables and general sequence of calculations for the standard shot. Conditions for the standard shot are;

- A target range of 50 yards;
- A bow height of 5 feet above ground;
- An arrow which, missing high, passes over target the target at a height of 5 feet above the ground; and,
- The archer, the target and the point at which arrow strikes ground are all on a level surface.

As previously noted in the Guide I present three sets of figures describing the actions of bows firing 180 ft/s (longbows and recurves), 240 ft/s (compounds) and 300 ft/s (extreme cam compounds). This was done to show the different capabilities of each type of bow. Also, I concentrated on data based on 50 yard target ranges to show requirements necessary to support up to the fastest bows and most distant targets. Unless a course is purpose designed for and stringently restricted to the slower bows, any course development must be designed to accommodate the faster (300 ft/s) bows.

THE FORMULAE

THE "STANDARD SHOT"
Variables and Calculations



Progression of calculations:

a. angle "α" from range "r" (see rquation 3)
b. height "h" from angle "α" (see equation 1)
c. height "H" is "h + 5"
d. ground angle "θ" from height "H" (see equation 4)
e. range "R" from angle "θ" (see equation 2)
f. distance "ρ" is 1/2 of "r" plus 1/2 of "R"

Figure B1 - The Standard Shot - Variables and Calculations

The basic equations I started with to calculate range data are the ballistic equations for a projectile's maximum altitude and range. These equations were taken from the text Calculus and Analytic Geometry, Pt II, 5th Edition, 1979, Thomas/Finney. In the equations which require it, I have used a value of 32.172 feet/second (980.616 m/s for the metrically inclined) for the acceleration due to gravity (**g**), this value is for sea level at 45° of latitude.

Projectile Maximum Altitude. The maximum altitude for a projectile is equal to the initial velocity (**V_o**) squared times the sine of the angle of departure (**α**) squared divided by two times the acceleration due to gravity (**g**).

$$h = \frac{(V_o \sin \alpha)^2}{2\,g}$$

EQUATION 1 – MAXIMUM ALTITUDE

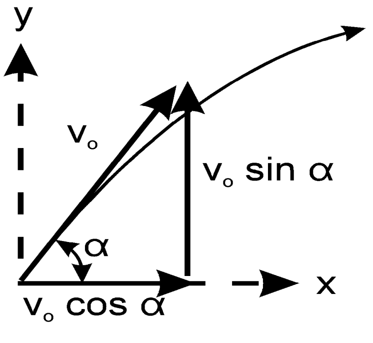
Projectile Range. The range of a projectile is equal to the initial velocity squared divided by the acceleration due to gravity and multiplied by the sine of twice the angle of departure.

$$r = \left(\frac{V_o^2}{g} \right) \sin 2\alpha$$

EQUATION 2 – RANGE

Angle of Departure from the Range. The above equations let us calculate the range and maximum altitude of a projectile if we know the angle of departure (launch) and the arrow velocity. However, the angle of departure at the bow was initially unknown. Based on the conditions I chose for the standard shot, the range to the point on the trajectory level with the point of launch was known, i.e.; the distance to the point five feet above the target through which the arrow passed is the target range. I also knew, of course, the selected initial velocities. I rearranged the above equation for range to develop an equation for the angle of departure based on the range.

Projectile Velocity Components



v_o = arrow velocity
α = angle of departure
vo sin α = vertical velocity
vo cos α = horizontal velocity
x = horizontal travel of arrow
(see equation 5)
y = vertical travel of arrow
(see equation 6)

Figure B2 - Projectile velocity components

Angle of Departure Based on Maximum Altitude. Similarly, I reworked the equation for maximum altitude to derive the angle of departure.

$$\boxed{\alpha = \mathbf{arcsin}\left(\frac{\sqrt{2\ g\ h}}{V_o}\right)}$$

Equation 4 – Angle of Departure based on Maximum Altitude

THE CALCULATIONS

Range to Impact for the Standard Shot. The four equations presented above formed the basis for all of the standard shot range data I calculated. In general terms, the following sequence of calculations was used to derive the standard shot distances:

A. The target range of 50 yards was used to calculate the angle of departure at the bow (Equation 3);

B. The maximum altitude above the five foot line (bow to level point above target) was calculated (Equation 1);

C. The maximum altitude above the five foot line plus the five feet above ground level of the launch point, was used to calculate a theoretical angle of departure (Equation 4). This was the angle of departure as if the arrow had been launched from ground level behind the archer to follow the same trajectory;

D. The theoretical ground angle of departure was used to calculate the maximum range for the arrow from ground level to ground level (Equation - 2); and,

$$\boxed{\alpha = \left(\frac{1}{2}\right)\ \mathbf{arcsin}\left(\frac{r\ g}{V_o^2}\right)}$$

Equation 3 – Angle of Departure based on Range

E. The distance from the archer to the point of impact at ground level was calculated by adding one-half the target range to one-half the theoretical range for the ground level launched trajectory. These would be the distances before and after the highest point on the trajectory, respectively.

See Table I for the distances to impact for the standard shot. Arrow velocities from 140 to 300 ft/s are included in the table. A second part to Table I shows the affect on range if the standard shot were projected over a target at 40, 30 or 20 yards.

The Ranges for Arrow Flight Over Falling Ground. The above sequence was repeated to calculate the distances to impact for cases where the ground falls away behind the target. For these calculations the theoretical ground angle of departure and ground level range were calculated based on the assumption that "ground level" was lower by the height of the drop in ground level behind the target. Drops of 10, 15 and 20 feet are examined for the three standard shot arrow velocities. See Table II for the distances to impact for the standard shot where the ground level drops behind the target.

Maximum Range With Changing Angle of Departure. This set of calculations was a departure from the standard shot as a test-bed. Starting with selected velocities and angles of departure, I made the following calculations;

A. The maximum altitude above the five foot bow height was calculated (Equation 1);

B. The maximum altitude above the five foot line, added to the five feet above ground level of the launch point, was used to calculate a theoretical angle of departure (Equation 4). This was the angle of departure as if the arrow had been launched from ground level behind the archer to follow the same trajectory;

C. The theoretical ground angle of departure was used to calculate the maximum range for the arrow from ground level to ground level (Equation - 2);

D. The distance the arrow would travel before reaching the five foot height of the bow launch point was derived by trial and error based on Equations 5 and 6;

E. Equations 5 and 6 use the initial velocity of a projectile ($\mathbf{V_o}$), its angle of departure($\boldsymbol{\alpha}$) and the time in flight (t) to derive the height attained ($\mathbf{y}$) and distance travelled ($\mathbf{x}$). The time value was adjusted until "x" equalled five feet (the bow height) and the corresponding value for "y" was calculated;

$$\boxed{x = (V_o \cos \alpha)\, t}$$

Equation 5 – Horizontal Movement of Projectile

$$\boxed{y = -\left(\frac{1}{2}\right) g\, t^2 + (V_o \sin \alpha)\, t}$$

Equation 6 – Vertical Movement of Projectile

F. The distance from the archer to the point of impact at ground level was calculated by subtracting the distance the arrow theoretically travelled to reach the height of the bow from the maximum range for a trajectory from ground level to ground level.

See Table III for the distances to impact for selected angles of departure and arrow velocities of 180, 240, and 300 ft/s.

Kinetic Energy. Although I have not used arrow kinetic energy data in this Guide, I decided to examine this equation and include it here. The kinetic energy (**KE**) of a moving object is equal to one-half its mass (**m**) multiplied by the square of its velocity (**v**), see equation 7. As the common unit of measure for arrow weight is the grain, we must insert conversion factors in the equation.

- One pound equals 7000 grains.
- The mass of an object is equal to its weight (**W**) divided by the acceleration due to gravity (**g**), which at sea level and 45 degrees of latitude is 32.172 feet per second squared.

Equation 8 shows the KE equation with the conversion factors from grain weight (W_{gr}) to mass inserted. This equation calculates the kinetic energy (**KE**) of an arrow in foot-pounds from the velocity (V_o) in feet per second and arrow weight in grains (W_{gr}).

$$\boxed{KE = \left(\frac{1}{2}\right) mv^2}$$

EQUATION 7 – KINETIC ENERGY

Conversion factors permit a direct calculation between the different units of measure.

$$\boxed{KE = \frac{W_{gr} V^2}{2 \times 7000 \times 32.172}}$$

EQUATION 8 – KINETIC ENERGY; GRAIN WEIGHT TO FOOT-POUNDS

Kinetic energy is a good means of comparing bows other than velocity alone. The following table for a brief comparison of bow kinetic energies when draw weight is increased and arrow weight is decreased to increase speed. In both presented cases, no appreciable change in arrow kinetic energy was realized. See Table IV for kinetic energy data for a wide range of arrow weights and velocities.

$$\boxed{KE = \frac{W_{gr} V^2}{450408}}$$

EQUATION 9 – ARROW KINETIC ENERGY

bow	draw weight	arrow weight	velocity	kinetic energy
	(lbs)	(gr)	(ft/s)	(ft-lbs)
High Country Ultra Extreme	55	550	222	60
	60	375	270	61
	Source for weights and velocities: Bowhunting World, Equipment Guide '92			
Martin's Ted Nugent Pro Safari	55	550	204	51
	60	375	248	51
	Source for weights and velocities: Bowhunting World, Equipment Guide '93			

THE TABLES

The Tables. The tables on the following pages were developed using Lotus 1-2-3 software. One aspect of using computers for this type of work is that calculations can be made with infinite precision. However, real life limitations such as air resistance and winds cannot be factored in easily, therefore calculated data are an approximation and digits beyond the decimal are mathematical mirage, not fact. For this reason, final range data is given in whole numbers. Even these must be balanced against all of the other factors covered in the Guide. The figures present a starting range which a course designer can then adjust based on actual angles of fire (upwards/downwards), target ranges, the lay of the land, obstacles, etc. In some cases in the tables I did not round off all figures to assist those who might wish to repeat my calculations.